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1999 J. Phys. A: Math. Gen. 32 7375

(http://iopscience.iop.org/0305-4470/32/42/310)

J. Phys. A: Math. Gen. **32** (1999) 7375–7392. Printed in the UK PII: S0305-4470(99)04295-X

Dynamics of the quantized radiation field in an oscillating cavity in the harmonic resonance case

Xiao-xue Yang† and Ying Wu†‡

† Physics Department, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

‡ Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, People's Republic of China

Received 14 May 1999, in final form 5 August 1999

Abstract. We investigate the dynamics of the quantized radiation field in an oscillating cavity described by the effective resonance Hamiltonians $H = \Omega\sum_{k=1}^{\infty} ka_k^\dagger a_k + \epsilon\Omega\sum_{k=1}^{\infty}\sqrt{k(k+j)}[a_k^\dagger a_{k+j} + a_{k+j}^\dagger a_k] + \epsilon\Omega H^{(p)}$ where ϵ characterizes the oscillating amplitude of the moving boundary of the cavity, the boundary oscillates with the jth unperturbed eigenfrequency ($j = 1, 2, 3, \ldots$) and the parametric oscillator part $H^{(p)}$ contains only a few terms to be specified in the main text. We present the exact diagonalization forms of the effective Hamiltonians for $j = 1, 2, 3, \ldots$ in the absence of the parametric oscillator part $H^{(p)}$. A systematic procedure is then developed to obtain the analytical solutions of the Hamiltonians in the presence of the part $H^{(p)}$ accurate up to order ϵ^k for an arbitrary positive integer k. In this way, we can investigate the dynamics of the corresponding quantized radiative field by explicitly presenting the analytical expressions of the diagonalized Hamiltonians, the time-varying annihilation and creation operators as well as photon number operators for the radiative field, accurate up to any desired order of the dimensionless oscillating amplitude. The analytical expressions of these quantities up to order ϵ^4 are also explicitly presented.

1. Introduction

Over the last two decades there has been intensive study focused on the quantum theory of the electromagnetic field in a cavity with moving boundaries [1–9]. Such a study is of fundamental theoretical interest in that it reveals a number of delicate features of quantum physics such as the dynamical modification of the Casimir force [3], the vacuum emission of photons with non-classical photon statistics [4–7] and the preparation of Schrödinger-cat states [8]. On the other hand, the subject is also of practical importance since it is closely related to sonoluminescence [9], high-precision optical interferometers [10], the generation of squeezed light [11] and quantum non-demolition measurements [12], etc. The quantization of the electromagnetic field in a cavity with time-varying boundaries is usually done in two different ways. One method is to construct the field operators directly from the solutions of the classical wave equation, and the Hamiltonian plays no role in the theory [1, 5]. It is therefore not possible to know the explicit form of the state of the field [6]. Another way, without this drawback, is to first derive an effective Hamiltonian for the subsequent study of the dynamics [6, 7]. The explicit form of the state of the field can then be known in principle in the latter formalism, and, once known, it provides a convenient basis for subsequent investigations of both the field's statistics and the resonant emission and absorption

0305-4470/99/427375+18$30.00

of photons by an atom placed in an oscillating cavity. Unfortunately, however, the derived Hamiltonians are usually too complicated to allow one to obtain the explicit analytical form of the state of the field. For instance, in investigating the field quantization and the effective Hamiltonian formalism, Law [6] considered a one-dimensional cavity formed by two perfectly reflecting mirrors with one mirror fixed at the position $x = 0$ and the other allowed to move in a prescribed trajectory $x = q(t)$. Taking $q(t) = L\exp[q_0\cos(j\omega t)/L]$ where $\omega = \pi/L$, $j = 1, 2, 3, \ldots$, and using the rotating-wave approximation, he explicitly wrote down the effective Hamiltonian for $j = 1, 2, 3$ and claimed that the complicated form in all three cases forbids one from finding the analytic solutions [6]. As a matter of fact, no one has, to our knowledge, succeeded in solving, even up to the second order of the dimensionless oscillating amplitude q_0/L, any of the quantized models described by these and other derived effective Hamiltonians in the resonance cases. In addition, it appears to be a very difficult task to develop a simple and systematical method to solve the quantized models described by the effective Hamiltonians even approximately because they involve infinite equally important interaction terms. This greatly hinders the subsequent investigation of both the field statistics and the resonant emission and absorption of photons by an atom placed in an oscillating cavity. It is therefore important and desirable to develop a method to accomplish it.

The resonance effective Hamiltonians derived by Law [6] have the form $H = H_0 + H_{int}$ with

$$H_0 = \omega I_0\left(\frac{q_0}{L}\right)\sum_{k=1}^{\infty} k a_k^\dagger a_k \tag{1.1a}$$

$$H_{int} = \frac{q_0\pi}{4L^2}\sum_{k=1}^{\infty}\sqrt{k(k+j)}\left[a_k^\dagger a_{k+j} + a_{k+j}^\dagger a_k\right] + \bar{H}^{(p)} \tag{1.1b}$$

where a_k and $a_k^\dagger$ are, respectively, the annihilation and creation operators for the kth mode of the radiation field, $j = 1, 2, 3, \ldots$, I_0 is the modified Bessel function of order zero, the first part in H_{int}, the summation with infinite terms in the right-hand side of equation (1.1b), describes the scattering-type interactions, and the parametric oscillator part $\bar{H}^{(p)}$ contains only a few terms and has a different form for different j. For example, $\bar{H}^{(p)} = 0$ in the first resonance case ($j = 1$), $\bar{H}^{(p)} = [q_0\pi/(4L^2)](a_1^2 + a_1^{\dagger 2})$ in the second resonance case ($j = 2$), while $H^{(p)} = \sqrt{2}[q_0\pi/(4L^2)](a_1a_2 + a_1^\dagger a_2^\dagger)$ in the third resonance case ($j = 3$). In writing equation (1.1), we [13] have utilized the simplified expression $f_\alpha(k) = \sqrt{k(k+\alpha)}$ for the function $f_\alpha(k) = k(k+\alpha)(2k+\alpha)^{-1}[\sqrt{(k+\alpha)/k} + \sqrt{k/(k+\alpha)}]$ in equation (3.7) of [6] (these two expressions of $f_\alpha(k)$ have been shown to be identical to each other in [13]), and here have supplied the explicit expression for the 'free' part H_0 which can be derived from equation (3.2) in [6] by using the rotating-wave approximation [13]. Recently [13], we have developed a method to solve the quantized model exactly in the first resonance case ($j = 1$) and have obtained explicit analytical expressions for the diagonalized Hamiltonian, the time-varying annihilation, creation and photon number operators for the radiative field in this case. Here we shall deal with the same problem except that here we consider the harmonic resonance situations $j = 2, 3, 4, \ldots$. Apart from different scattering terms, the essential difference between the Hamiltonian in the fundamental resonance case and those in the harmonic resonance situations is that the former does not have a parametric oscillator part, while the latter Hamiltonians contain this part. However, it will be found that the introduction of a parametric oscillator part in harmonic resonance cases increases considerably the complexity of the diagonalization.

In this paper, it will be shown that we can exactly diagonalize the effective Hamiltonians for any harmonic resonance case in the absence of the parametric oscillator part with the same spirit as we did in the fundamental resonance case except that unlike in the fundamental case, j different unitary operators are needed to relate the bare photonic operators to the corresponding dressed ones in the jth harmonic resonance situation. We then develop a systematic method to deal with the problem in the presence of the parametric oscillator part. It is shown that we can obtain analytic solutions accurate up to the kth order of the dimensionless oscillating amplitude of the moving mirror (here k can be any positive integer, i.e. $k = 1, 2, 3, \ldots$). In this way, we can investigate the dynamics of the corresponding quantized radiative field by explicitly presenting the analytical expressions of the diagonalized Hamiltonians, the time-varying annihilation and creation operators as well as photon number operators for the radiative field accurate up to any desired order of the dimensionless oscillating amplitude. In particular, we shall present the explicit analytical expressions of the above-mentioned operators accurate up to the fourth order of the dimensionless oscillating amplitude of the moving mirror in the second harmonic case. It is worthwhile to mention that the previous studies on a one-dimensional cavity with an oscillating mirror in the resonance or near-resonance cases are usually accurate only up to the first order of the dimensionless oscillating amplitude [6, 7]. This paper is organized as follows. In section 2, we diagonalize explicitly the effective Hamiltonians in the absence of parametric oscillator terms without any approximation for all harmonic resonance cases. In section 3, to deal with the diagonalization in the presence of the parametric oscillator part, we develop a systematic method and the corresponding formulae to express analytical solutions accurate up to the kth order of the dimensionless oscillating amplitude of the moving mirror (here k can be any positive integer, i.e. $k = 1, 2, 3, \ldots$). In section 4, we investigate the corresponding dynamics by obtaining explicitly the analytical expressions of time-varying annihilation, creation and photon number operators, and section 5 concludes the paper with some discussions.

2. Diagonalization of the Hamiltonians in the absence of $H^{(p)}$

The effective Hamiltonians in equation (1.1) can be rewritten as $H = H' + \Omega H^{(p)}$ with

$$\frac{H'}{\Omega} = \sum_{k=1}^{\infty} k a_k^\dagger a_k + \epsilon \sum_{k=1}^{\infty} \sqrt{k(k+j)} \left[a_k^\dagger a_{k+j} + a_{k+j}^\dagger a_k \right] \tag{2.1}$$

where $\Omega = \omega I_0(q_0/L)$, $\omega = \pi/L$ is the fundamental frequency of the unperturbed cavity with a fixed length L, I_0 is the modified Bessel function of order zero, $\epsilon = q_0/[4LI_0(q_0/L)]$ characterizing the dimensionless oscillating amplitude of the moving mirror, $j = 1, 2, 3, 4, \ldots$ describe the jth harmonic resonance case where the moving mirror oscillates with the jth unperturbed eigenfrequency $j\omega$. Note that the maximum value of the small parameter ϵ is $\epsilon_{max} \approx 0.23$ for any q_0/L. The parametric oscillator part $H^{(p)}$ becomes $H^{(p)} = \epsilon(a_1^2 + a_1^{\dagger 2})$ in the second resonance case ($j = 2$), while $H^{(p)} = \epsilon\sqrt{2}(a_1 a_2 + a_1^\dagger a_2^\dagger)$ in the third resonance case ($j = 3$).

In this section, we show explicitly that the Hamiltonian H' in equation (2.1) can be diagonalized exactly for any resonance case. We begin with the second resonance case ($j = 2$) to illustrate the diagonalization method, and then generalize it to include other harmonic resonance cases as well.

2.1. Second resonance case

Equation (2.1) for the second resonance case ($j = 2$) can be put into another form: $H' = H_{even} + H_{odd}$ with

$$\frac{H_{even}}{\Omega} = 2\sum_{k=1}^{\infty} k a_{2k}^{\dagger} a_{2k} + 2\epsilon \sum_{k=1}^{\infty} \sqrt{k(k+1)}\big[a_{2k}^{\dagger} a_{2(k+1)} + a_{2(k+1)}^{\dagger} a_{2k}\big] \tag{2.2a}$$

$$\frac{H_{odd}}{\Omega} = \sum_{k=0}^{\infty}(2k+1) a_{2k+1}^{\dagger} a_{2k+1} + \epsilon \sum_{k=0}^{\infty} \sqrt{(2k+1)(2k+3)}\big[a_{2k+1}^{\dagger} a_{2k+3} + a_{2k+3}^{\dagger} a_{2k+1}\big]. \tag{2.2b}$$

We introduce a fictitious harmonic oscillator described by the annihilation and creation operators A and $A^{\dagger}$ as well as the corresponding number operator $N = A^{\dagger}A$. The operators A and $A^{\dagger}$ satisfy the usual commutation relation $[A, A^{\dagger}] = 1$, and they commute with all the operators a_k and $a_k^{\dagger}$ of the radiation field. Let $|n\rangle$, $n = 0, 1, 2, \ldots$ denote the eigenkets of the number operator $N = A^{\dagger}A$ and use the relations $A|n\rangle = \sqrt{n}|n-1\rangle$ and $A^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$, we can rewrite the 'even' and 'odd' part of the effective Hamiltonian as follows:

$$\frac{H_{even}}{2\Omega} = \sum_{n,m=1}^{\infty} a_{2n}^{\dagger}\langle n|[N + \epsilon(\sqrt{N}A + A^{\dagger}\sqrt{N})]|m\rangle a_{2m} \tag{2.3a}$$

$$\frac{H_{odd}}{\Omega} = \sum_{n,m=0}^{\infty} a_{2n+1}^{\dagger}\langle n|[(2N+1) + \epsilon(f_N A + A^{\dagger} f_N)]|m\rangle a_{2m+1} \tag{2.3b}$$

where $f_N = \sqrt{(2N+1)(2N+3)/(N+1)}$. It is easy to show

$$\sqrt{1-4\epsilon^2}\,U(\theta) N U(-\theta) = N + \epsilon(\sqrt{N}A + A^{\dagger}\sqrt{N}) \tag{2.4a}$$

$$\sqrt{1-4\epsilon^2}\,V(\phi)(2N+1)V(-\phi) = (2N+1) + \epsilon(f_N A + A^{\dagger} f_N) \tag{2.4b}$$

where

$$U(\theta) = \exp[\theta(\sqrt{N}A - A^{\dagger}\sqrt{N})] \tag{2.5a}$$

$$V(\phi) = \exp[\phi(f_N A - A^{\dagger} f_N)] \tag{2.5b}$$

where $\phi = \theta/2 = \tanh^{-1}(2\epsilon)/4$ and $\epsilon = q_0/[4LI_0(q_0/L)]$. Both $U(\theta)$ and $V(\phi)$ are unitary operators so that $U^{-1}(\theta) = U^{\dagger}(\theta) = U(-\theta)$ and $V^{-1}(\phi) = V^{\dagger}(\phi) = V(-\phi)$. Substituting equation (2.4) into equation (2.3), and utilizing the completeness relation $\sum_{k=0}^{\infty}|k\rangle\langle k| = 1$, and $U_{n0}(\theta) = U_{0l}(-\theta) = 0$, it is straightforward to obtain

$$\frac{H_{even}}{2\Omega} = \sqrt{1-4\epsilon^2} \sum_{n,m,l,k=1}^{\infty} a_{2n}^{\dagger}\langle n|U(\theta)|l\rangle\langle l|N|k\rangle\langle k|U(-\theta)|m\rangle a_{2m} \tag{2.6a}$$

$$\frac{H_{odd}}{\Omega} = \sqrt{1-4\epsilon^2} \sum_{n,m,l,k=0}^{\infty} a_{2n+1}^{\dagger}\langle n|V(\phi)|l\rangle\langle l|(2N+1)|k\rangle\langle k|V(-\phi)|m\rangle a_{2m+1}. \tag{2.6b}$$

We introduce a new set of photonic operators b_n, $b_n^{\dagger}$, $n = 1, 2, \ldots$ for the radiation field by the unitary transformation,

$$b_{2n} = \sum_{m=1}^{\infty} U_{nm}(-\theta) a_{2m} \qquad a_{2m} = \sum_{n=1}^{\infty} U_{mn}(\theta) b_{2n} \tag{2.7a}$$

$$b_{2k+1} = \sum_{j=0}^{\infty} V_{kj}(-\phi) a_{2j+1} \qquad a_{2j+1} = \sum_{k=0}^{\infty} V_{jk}(\phi) b_{2k+1} \tag{2.7b}$$

where the matrix elements are defined by $U_{nm}(\theta) = \langle n|U(\theta)|m\rangle$ and $V_{jk}(\phi) = \langle j|V(\phi)|k\rangle$, and they satisfy $U_{mn}^{*}(\theta) = U_{nm}^{\dagger}(\theta) = U_{nm}^{-1}(\theta) = U_{nm}(-\theta)$ and $V_{jk}^{*}(\phi) = V_{kj}^{\dagger}(\phi) = V_{kj}^{-1}(\phi) =$

$V_{kj}(-\phi)$. From the definition of the operators b_n, $b_n^\dagger$, $n = 1, 2, \ldots$, one easily finds that they satisfy the relations $[b_n, b_m] = [b_n^\dagger, b_m^\dagger] = 0$, and $[b_n, b_m^\dagger] = \delta_{nm}$ by utilizing the counterparts for the operators a_n,$a_n^\dagger$, $n = 1, 2, \ldots$, and the properties of the unitary operators U and V.

Using equations (2.6) and (2.7), and the properties $\langle l|N|k\rangle = k\delta_{lk}$, $U_{mn}^*(\theta) = U_{nm}^\dagger(\theta) = U_{nm}(-\theta)$ and $V_{jk}^*(\phi) = V_{kj}^\dagger(\phi) = V_{kj}(-\phi)$, we arrive at $H_{even} = \Omega\sqrt{1-4\epsilon^2}\sum_{k=1}^\infty 2kb_{2k}^\dagger b_{2k}$ and $H_{odd} = \Omega\sqrt{1-4\epsilon^2}\sum_{k=0}^\infty (2k+1)b_{2k+1}^\dagger b_{2k+1}$ or

$$H' = \Omega\sqrt{1-4\epsilon^2}\sum_{k=1}^\infty kb_k^\dagger b_k \tag{2.8}$$

where $\Omega = \omega I_0(q_0/L)$, $\omega = \pi/L$ is the fundamental eigenfrequency of the unperturbed cavity and $\epsilon = q_0/[4LI_0(q_0/L)]$. This equation, together with the photonic operators $b_n, n = 1, 2, 3, \ldots$ determined by equations (2.5) and (2.7), gives explicitly the diagonalized form of the effective Hamiltonian in the absence of the parametric oscillator term $H^{(p)}$ in the second resonance case.

2.2. *General resonance cases*

In this subsection, we illustrate that the Hamiltonian H' in equation (2.1) for any other harmonic resonance cases ($j = 3, 4, 5, \ldots$) can also be diagonalized by the same method and its diagonalized form for any positive integer j is still given by equation (2.8). For a fixed j ($j = 1, 2, 3, \ldots$), H' in equation (2.1) can be expressed as $H' = \sum_{l=1}^j H_j^{(l)}$ with

$$\frac{H_j^{(j)}}{\Omega} = j\sum_{n=1}^\infty\left[na_{nj}^\dagger a_{nj} + \sqrt{n(n+1)}\left(a_{nj}^\dagger a_{(n+1)j} + a_{(n+1)j}^\dagger a_{nj}\right)\right] \tag{2.9a}$$

$$\frac{H_j^{(k)}}{\Omega} = \sum_{n=0}^\infty\left[(nj+k)a_{nj+k}^\dagger a_{nj+k} + \sqrt{(nj+k)[(n+1)j+k]}\left(a_{nj+k}^\dagger a_{(n+1)j+k} + a_{(n+1)j+k}^\dagger a_{nj+k}\right)\right] \tag{2.9b}$$

where $k = 1, 2, \ldots, (j-1)$. Note that $H' \equiv H_1^{(1)}$ for $j = 1$. After some manipulations, we can obtain

$$\begin{aligned}\frac{H_j^{(j)}}{\Omega} &= j\sum_{n,m=1}^\infty a_{nj}^\dagger\langle n|[N + \epsilon(\sqrt{N}A + A^\dagger\sqrt{N})]|m\rangle a_{mj}\\ &= j\sqrt{1-4\epsilon^2}\sum_{n,m=1}^\infty a_{nj}^\dagger\langle n|U(\theta)NU(-\theta)|m\rangle a_{mj}\end{aligned} \tag{2.10a}$$

$$\begin{aligned}\frac{H_j^{(k)}}{\Omega} &= \sum_{n,m=0}^\infty a_{nj+k}^\dagger\langle n|[(Nj+k) + \epsilon(f_N^{j;k}A + A^\dagger f_N^{j;k})]|m\rangle a_{mj+k}\\ &= \sqrt{1-4\epsilon^2}\sum_{n,m=0}^\infty a_{nj}^\dagger\langle n|U^{(k)}(\psi_j)(Nj+k)U^{(k)}(-\psi_j)|m\rangle a_{mj}\end{aligned} \tag{2.10b}$$

where $f_N^{j;k} = \sqrt{(Nj+k)[(N+1)j+k]/(N+1)}$, $\psi_j = \tanh^{-1}(2\epsilon)/2j$, and $U^{(k)}(\psi_j) = \exp[\psi_j(f_N^{j;k}A - A^\dagger f_N^{j;k})]$, $k = 1, 2, \ldots, (j-1)$. Here we have made use of the relations $\sqrt{1-4\epsilon^2}\,U^{(k)}(\psi_j)(Nj+k)U^{(k)}(-\psi_j) = [(Nj+k) + \epsilon(f_N^{j;k}A + A^\dagger f_N^{j;k})]$ which are proved in appendix A. Therefore, we can introduce a new set of photonic operators $\{b_n, b_n^\dagger; n = 1, 2, \ldots\}$ by the unitary transformation,

$$b_{nj} = \sum_{m=1}^{\infty} U_{nm}(-\theta)\, a_{mj} \qquad a_{mj} = \sum_{n=1}^{\infty} U_{mn}(\theta)\, b_{nj} \tag{2.11a}$$

$$b_{nj+k} = \sum_{m=0}^{\infty} U_{nm}^{(k)}(-\psi_j)\, a_{mj+k} \qquad a_{mj+k} = \sum_{n=0}^{\infty} U_{mn}^{(k)}(\psi_j)\, b_{nj+k}. \tag{2.11b}$$

Following the same routine as in the last subsection, we obtain

$$H_j^{(j)} = \Omega\sqrt{1-4\epsilon^2}\sum_{n=1}^{\infty} nj b_{nj}^\dagger b_{nj} \tag{2.12a}$$

$$H_j^{(k)} = \Omega\sqrt{1-4\epsilon^2}\sum_{n=0}^{\infty}(nj+k) b_{nj+k}^\dagger b_{nj+k} \qquad k = 1, 2, \ldots, (j-1). \tag{2.12b}$$

These results, together with $H' = H_j^{(k)} + \sum_{k=1}^{j-1} H_j^{(k)}$ ($H' \equiv H_1^{(1)}$ for $j = 1$), immediately lead to $H' = \Omega\sqrt{1-4\epsilon^2}\sum_{k=1}^{\infty} k b_k^\dagger b_k$, i.e. equation (2.8) is the diagonalized form of the Hamiltonians H' in equation (2.1) for any positive integer j.

In this section, we have shown explicitly that the Hamiltonians H' in equation (2.1) for any positive integer j can be diagonalized exactly and they have the identical diagonalized form although for different positive integers j, different numbers (j) and forms of unitary operators are needed to establish the relation between two sets of the photonic operators $\{a_n, a_n^\dagger; n = 1, 2, \ldots\}$ and $\{b_n, b_n^\dagger; n = 1, 2, \ldots\}$. The operators $a_n, a_n^\dagger, n = 1, 2, 3, \ldots$ describe the bare photons of the radiation field, while $b_n, b_n^\dagger, n = 1, 2, 3, \ldots$ describe in some sense the corresponding dressed photons due to the scattering-type interactions caused by the mirror oscillations. In other words, in a cavity with fixed boundaries, there exists no any interaction among the bare photons of the radiation field described by operators $a_n, a_n^\dagger, n = 1, 2, 3, \ldots$. However, these bare photons will have interactions among themselves in the presence of the mirror oscillations as shown in equation (1). In the latter case, the effect of the scattering-type interactions due to the mirror oscillations is to cause the dressing of bare photons into dressed ones described by $b_n, b_n^\dagger, n = 1, 2, 3, \ldots$. As will be seen shortly, the introduction of the dressed photons described by $b_n, b_n^\dagger$ considerably simplifies the discussion of the role played by the interactions due to the parametric oscillator part. Such an approach of introducing dressed particles is frequently seen in the fields of quantum optics [15] and condensed physics. For instance, one way to deal with the system of a bare atom subject to two quantized laser beams (called signal and probe beams, respectively) is to introduce a dressed atom (bare atom plus the signal beam) and consider the interaction of the dressed atom with the probe beam [15]. One effect of the mirror oscillations is to cause strong scattering-type couplings among the bare photons of different modes of the radiation cavity field, but there exists no such kind of interaction among the dressed photons of different modes. Note that in the absence of the parametric oscillator term $H^{(p)}$, the diagonalized Hamiltonian in any jth resonance case is identical in form to any other and to the one describing the radiation field in a cavity with fixed mirrors and length $L' = \pi/\Omega = L/\sqrt{I_0^2(q_0/2L) - (q_0/2L)^2}$.

3. Diagonalization in the presence of $H^{(p)}$

In this section, we consider the diagonalization of the effective Hamiltonian in the presence of the parametric oscillator part $H^{(p)}$. The parametric oscillator part $H^{(p)}$ has a different form for different harmonic resonance cases. In the second resonance case ($j = 2$), $H^{(p)} = \epsilon(a_1^2 + a_1^{\dagger 2})$ is expressed in terms of the dressed photonic operators $b_n, b_n^\dagger$ by using $a_1 = \sum_{k=0}^{\infty} V_{0k}(\phi)\, b_{2k+1}$ in equation (2.7) as follows:

$$H^{(p)} = \epsilon \sum_{n,m=0}^{\infty} B_{nm}\big(b_{2n+1}b_{2m+1} + b_{2m+1}^\dagger b_{2n+1}^\dagger\big) \qquad j = 2 \tag{3.1}$$

where B is a real symmetric matrix with its matrix elements $B_{nm} = \phi_n\phi_m$ and $\phi_n \equiv V_{0n}(\phi)$. While $H^{(p)} = \epsilon\sqrt{2}(a_1a_2 + a_1^\dagger a_2^\dagger)$ in the third resonance case ($j = 3$), by using $a_k = \sum_{n=0}^{\infty} U_{0n}^{(k)}(\psi_3) b_{3n+k}$, $k = 1, 2$ in equation (2.11), becomes

$$H^{(p)} = \sqrt{2}\epsilon \sum_{n,m=0}^{\infty} E_{nm}\big(b_{3n+1}b_{3m+2} + b_{3m+2}^\dagger b_{3n+1}^\dagger\big) \qquad j = 3 \tag{3.2}$$

where $E_{nm} = U_{0n}^{(1)}(\psi_3)\, U_{0m}^{(2)}(\psi_3)$ are also real numbers. Here we have made use of the fact that all the matrix elements of the unitary transformation operators $V_{nm}(\phi)$, $U_{nm}^{(k)}(\psi_3)$ are real numbers. This fact is the direct result of the explicit expressions of the unitary operators $V(\phi)$, $U^{(k)}(\psi_3)$ in the previous section.

It is noted from equations (3.1) and (3.2) that the dressed modes described by $\{b_{nj}, b_{nj}^\dagger; n = 1, 2, 3, \ldots\}$ are still non-interacting modes even in the presence of the parametric oscillator term because $H^{(p)}$ only causes coupling among those dressed modes described by $\{b_{nj+k}, b_{nj+k}^\dagger; k = 1, 2, \ldots, (j-1); n = 1, 2, 3, \ldots\}$. The parametric oscillator part $H^{(p)}$ contains only a few terms when it is expressed in terms of the bare photonic operators but it involves infinite terms when it is expressed in terms of the dressed photonic operators. However, the infinite terms originating from a few parametric oscillator terms differ from each other in the order of the small parameter ϵ because of $B_{nm} = \mathrm{O}\big(\epsilon^{n+m}\big)$, $E_{nm} = \mathrm{O}\big(\epsilon^{n+m}\big)$ (see appendix A), we do not need to deal with all of them simultaneously but just a small number of them for any specified order of the parameter ϵ. For instance, up to order ϵ^k, $H^{(p)}$ in equation (3.1) becomes $H^{(p)} = \epsilon \sum_{n,m=0}^{k-1} B_{nm}(b_{2n+1}b_{2m+1} + b_{2m+1}^\dagger b_{2n+1}^\dagger) + \mathrm{O}\big(\epsilon^{k+1}\big)$ contains k dressed modes.

In this section, we develop a systematic method to diagonalize the total Hamiltonian $H = H' + H^{(p)}$ up to order ϵ^k where k can be any positive integer. This method is suitable for all harmonic resonance cases, but here we shall focus on the second harmonic resonance case ($j = 2$) for clarity. Using equations (2.8) and (3.1), we can write the total Hamiltonian $H = H' + H^{(p)}$ for the second harmonic resonance case ($j = 2$) as follows:

$$H = \Omega\sqrt{1-4\epsilon^2}\left[\sum_{n=1}^{\infty} 2n b_{2n}^\dagger b_{2n} + \sum_{n=k}^{\infty}(2n+1) b_{2n+1}^\dagger b_{2n+1}\right] + H^{(k)} + \mathrm{O}\big(\epsilon^{k+1}\big) \tag{3.3}$$

with

$$\begin{aligned}\frac{H^{(k)}}{\Omega\sqrt{1-4\epsilon^2}} &= \sum_{n=0}^{k-1}(2n+1) b_{2n+1}^\dagger b_{2n+1} + \tfrac{1}{2}\eta \sum_{n,m=0}^{k-1} B_{nm}\big(b_{2n+1}b_{2m+1} + b_{2m+1}^\dagger b_{2n+1}^\dagger\big)\\ &= \tfrac{1}{2}\big(\ \tilde{\alpha}^{(\dagger)}, \tilde{\alpha}\ \big)\begin{pmatrix} F & \eta B \\ \eta B & F \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^{(\dagger)} \end{pmatrix} - \tfrac{1}{2}\sum_{n=0}^{k-1}(2n+1)\end{aligned} \tag{3.4}$$

where $\eta = 2\epsilon/\sqrt{1-4\epsilon^2}$, B and F are two $k \times k$ real matrices with the matrix elements $B_{nm} = \phi_n\phi_m$ and $F_{nm} = (2n+1)\delta_{nm}$, $n, m = 0, 1, 2, \ldots, (k-1)$, respectively, $\phi_n \equiv V_{0n}(\phi)$,

$\tilde{\alpha} = (b_1, b_3, b_5, \ldots, b_{2k-1})$, $\tilde{\alpha}^{(\dagger)} = (b_1^\dagger, b_3^\dagger, b_5^\dagger, \ldots, b_{2k-1}^\dagger)$, and

$$\alpha = \begin{pmatrix} b_1 \\ b_3 \\ \vdots \\ b_{2k-1} \end{pmatrix} \qquad \alpha^{(\dagger)} = \begin{pmatrix} b_1^\dagger \\ b_3^\dagger \\ \vdots \\ b_{2k-1}^\dagger \end{pmatrix}. \tag{3.5}$$

Equations (3.3) and (3.4) display that up to order ϵ^k (where k can be any positive integer), the dressed modes described by $\{b_{2n}, b_{2n}^\dagger, b_{2m+1}, b_{2m+1}^\dagger; m \geqslant k, n = 1, 2, \ldots\}$ are non-interacting modes and the diagonalization of the total Hamiltonian reduces to that of $H^{(k)}$ in equation (3.4). In order to diagonalize $H^{(k)}$, we introduce a set of new photonic operators $c_{2n+1}, c_{2n+1}^\dagger, n = 0, 1, 2, \ldots, (k-1)$ by the transformation

$$\begin{pmatrix} \beta \\ \beta^{(\dagger)} \end{pmatrix} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^{(\dagger)} \end{pmatrix} \qquad \begin{pmatrix} \alpha \\ \alpha^{(\dagger)} \end{pmatrix} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}^{-1} \begin{pmatrix} \beta \\ \beta^{(\dagger)} \end{pmatrix} \tag{3.6}$$

where P and Q are two $k\times k$ real matrices to be specified later, and $\beta = \alpha|_{b\to c}$, etc, for example, $\tilde{\beta} = (c_1, c_3, c_5, \ldots, c_{2k-1})$. The requirement that both $\{c_{2n+1}, c_{2n+1}^\dagger; n = 0, 1, \ldots, (k-1)\}$ and $\{b_{2n+1}, b_{2n+1}^\dagger; n = 0, 1, \ldots, (k-1)\}$ are photonic operator sets is easily shown to result in

$$PP^T - QQ^T = I \qquad PQ^T = QP^T \tag{3.7a}$$

$$P^TP - Q^TQ = I \qquad P^TQ = Q^TP \tag{3.7b}$$

where the superscript T denotes the transpose operation, and I is the $k \times k$ unit matrix. These equations are combined to give

$$\begin{pmatrix} P & Q \\ Q & P \end{pmatrix}^{-1} = \begin{pmatrix} P^T & -Q^T \\ -Q^T & P^T \end{pmatrix}.$$

Utilizing equations (3.6) and (3.7), equation (3.4) becomes

$$\frac{H^{(k)}}{\Omega\sqrt{1-4\epsilon^2}} = \tfrac{1}{2}(\beta^\dagger \quad \tilde{\beta}) \begin{pmatrix} P & -Q \\ -Q & P \end{pmatrix} \begin{pmatrix} F & \eta B \\ \eta B & F \end{pmatrix} \begin{pmatrix} P^T & -Q^T \\ -Q^T & P^T \end{pmatrix} \begin{pmatrix} \beta \\ \tilde{\beta}^\dagger \end{pmatrix} - \tfrac{1}{2}\sum_{n=0}^{k-1}(2n+1) \tag{3.8}$$

which is a diagonal form

$$H^{(k)} = \Omega\sqrt{1-4\epsilon^2}\sum_{n=0}^{k-1}\left\{\lambda_{2n+1}c_{2n+1}^\dagger c_{2n+1} + \tfrac{1}{2}[\lambda_{2n+1} - (2n+1)]\right\} \tag{3.9}$$

if we choose the two real matrices P and Q to satisfy the equation

$$\begin{pmatrix} P & -Q \\ -Q & P \end{pmatrix} \begin{pmatrix} F & \eta B \\ \eta B & F \end{pmatrix} \begin{pmatrix} P^T & -Q^T \\ -Q^T & P^T \end{pmatrix} = \begin{pmatrix} \Lambda & O \\ O & \Lambda \end{pmatrix} \tag{3.10}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_3, \ldots, \lambda_{2k-1})$ is a diagonal matrix and O is a $k \times k$ zero matrix. This equation is equivalent to two matrix equations $FP^T - \eta BQ^T = P^T\Lambda$ and $\eta BP^T - FP^T = Q^T\Lambda$ or $(F+\eta B)(P^T - Q^T) = (P^T + Q^T)\Lambda$ and $(F-\eta B)(P^T + Q^T) = (P^T - Q^T)\Lambda$. The latter two matrix equations combine to give the following eigenvalue equations:

$$(F+\eta B)(F-\eta B)R_+ = R_+\Lambda^2 \qquad (F-\eta B)(F+\eta B)R_- = R_-\Lambda^2 \tag{3.11}$$

where $R_{\mp} = (P^T \pm Q^T)$ are two real matrices satisfying, by means of equation (3.7), the relations $R_+^T R_- = R_+ R_-^T = I$. Here $\eta = 2\epsilon/\sqrt{1-4\epsilon^2}$, B and F are two $k \times k$ real matrices with the matrix elements $B_{nm} = \phi_n \phi_m$, $F_{nm} = (2n+1)\delta_{nm}$, $n, m = 0, 1, 2, \ldots, (k-1)$ and $\phi_n \equiv V_{0n}(\phi)$. Note that the diagonal matrix Λ is an even function of the variable η, i.e. $\Lambda(\eta) = \Lambda(-\eta)$. This property is easily seen from the first line of equation (3.4) by noting that the transformation $b_{2n+1} \to \mathrm{i}b_{2n+1}$, $n = 0, 1, \ldots$ should not alter the eigenvalues of $H^{(k)}$ but it amounts to changing the sign of the variable η. Therefore, we only actually need to solve one of the eigenvalue equations in equation (3.11), say $(F+\eta B)(F-\eta B)R_+ = R_+\Lambda^2$, to obtain all the k eigenvalues λ_{2n+1}, $n = 0, 1, \ldots, (k-1)$ and the matrix R_+. Once this is done, R_- is obtained simply by $R_- = R_+|_{\eta\to-\eta}$ and hence the required transformation matrices $P = (R_+^T + R_-^T)/2$, $Q = (R_+^T - R_-^T)/2$. Then the relations between two sets of photonic operators can be obtained by equations (3.6) and (3.7) or

$$\begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{2k-1} \end{pmatrix} = P \begin{pmatrix} b_1 \\ b_3 \\ \vdots \\ b_{2k-1} \end{pmatrix} + Q \begin{pmatrix} b_1^\dagger \\ b_3^\dagger \\ \vdots \\ b_{2k-1}^\dagger \end{pmatrix} \tag{3.12a}$$

$$\begin{pmatrix} b_1 \\ b_3 \\ \vdots \\ b_{2k-1} \end{pmatrix} = P^T \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{2k-1} \end{pmatrix} - Q^T \begin{pmatrix} c_1^\dagger \\ c_3^\dagger \\ \vdots \\ c_{2k-1}^\dagger \end{pmatrix} \tag{3.12b}$$

or the corresponding component forms

$$b_{2n+1} = \sum_{m=0}^{k-1} [P_{mn} c_{2m+1} - Q_{mn} c_{2m+1}^\dagger] \qquad n = 0, 1, \ldots \tag{3.13a}$$

$$c_{2m+1} = \sum_{n=0}^{k-1} [P_{mn} b_{2n+1} + Q_{mn} b_{2n+1}^\dagger] \qquad m = 0, 1, \ldots. \tag{3.13b}$$

The diagonal form of corresponding total Hamiltonian accurate up to $\mathrm{O}(\epsilon^k)$ is

$$\frac{H}{\Omega\sqrt{1-4\epsilon^2}} = \sum_{n=1}^{\infty} 2n b_{2n}^\dagger b_{2n} + \sum_{n=0}^{\infty} \lambda_{2n+1} c_{2n+1}^\dagger c_{2n+1} + \frac{1}{2}\sum_{n=0}^{k-1} [\lambda_{2n+1} - (2n+1)] + \mathrm{O}(\epsilon^k) \tag{3.14}$$

where $\lambda_{2n+1} = (2n+1)$, $c_{2n+1} = b_{2n+1}$ as $n \geqslant k$, while λ_{2n+1}, $n = 0, 1, \ldots, (k-1)$ are determined by equation (3.11), and the relation between $\{c_{2n+1}, c_{2n+1}^\dagger, n = 0, 1, \ldots, (k-1)\}$ and $\{b_{2n+1}, b_{2n+1}^\dagger, n = 0, 1, \ldots, (k-1)\}$ is given by equation (3.13).

We have shown that the diagonalization of the total Hamiltonian in the second harmonic resonance case up to order ϵ^k (where k can be any positive integer) reduces to solving the eigenvalue problem of a $k \times k$ matrix. Such a method is easily seen to be suitable for other harmonic resonance cases as well. As a matter of fact, for the fairly general bilinear k mode Hamiltonian of the form

$$H = \sum_{n=0}^{k-1} \omega_n b_n^\dagger b_n + \sum_{n,m=0}^{k-1} [(B_{nm} b_n^\dagger b_m + \mathrm{h.c.}) + (C_{nm} b_n^\dagger b_m^\dagger + \mathrm{h.c.})]$$

such a method with minor modifications is also suitable. It is worthwhile to mention that the reduction in this section is mainly based on the transformation introduced in the previous section. Without expressing the total Hamiltonian in terms of the dressed operators $\{b_n, b_n^\dagger; n = 1, 2, 3, \ldots\}$, the finite parametric oscillator terms and infinite scattering terms are of the same order of the small parameter ϵ, and one has to deal with the infinite terms simultaneously on an equal footing. Also, it is difficult to know how to handle simply and directly the infinite terms simultaneously on an equal footing accurate up to the higher order of the parameter ϵ, say, the ϵ^2 order. This partially accounts for why no one has yet succeeded in dealing analytically with the dynamics of the quantized field in an oscillating cavity in resonance cases even up to order ϵ^2. In contrast, it is now a simple matter to obtain results accurate up to ϵ^k for any positive integer k by solving the eigenvalue problem of a $k \times k$ matrix. In particular, it is straightforward to solve an eigenvalue problem of a 2×2 matrix to obtain the results up to ϵ^2. In appendix B, we have presented a systematic procedure to solve the eigenvalue equation (3.11) when k is any positive integer. Here we list the results accurate up to $\mathrm{O}(\epsilon^4)$ ($k = 4$). The eigenvalues are $\lambda_1 = \sqrt{1 - 4\epsilon^2 - 13\epsilon^4} + \mathrm{O}(\epsilon^6)$, $\lambda_3 = 3\sqrt{1 - \epsilon^4/2} + \mathrm{O}(\epsilon^6)$, $\lambda_{2n+1} = (2n+1) + \mathrm{O}(\epsilon^6)$ as $n \geqslant 2$, and the transformation matrices P and Q are listed in appendix B. The effect of the parametric oscillator term $H^{(p)}$ on the eigenvalues is $[\lambda_{2n+1} - (2n+1)], n = 0, 1, \ldots$. Except for λ_1, this effect is negligibly small even for a relatively large oscillating amplitude of the moving mirror. For instance, the relative eigenvalue shifts due to $H^{(p)}$ are $[\lambda_{2n+1} - (2n+1)]/(2n+1)$, for $n = 1$ this is approximately equal to $-\epsilon^4/4 \approx -0.0007$ even for the maximum $\epsilon_{max} \approx 0.23$ where $\epsilon = q_0/[4LI_0(q_0/L)]$. Consequently, the main effect of the parametric oscillator term $H^{(p)}$ is to cause photon creation and annihilation of the field modes.

4. The dynamics of the radiation field

In this section, we investigate the dynamics of the radiation field by explicitly giving the time-evolving expressions of the annihilation and creation operators of the field. We focus on the second harmonic resonance case. Using the explicit diagonal form of the total Hamiltonian in the second resonance case accurate up to $\mathrm{O}(\epsilon^k)$ in equation (3.14) and the Heisenberg equation $\mathrm{d}f/\mathrm{d}t = -\mathrm{i}[f, H]$, and noting that two operator sets $\{b_{2n}, b_{2n}^\dagger; n = 1, 2, \ldots\}$ and $\{c_{2m+1}, c_{2m+1}^\dagger; m = 0, 1, \ldots\}$ are independent of each other (i.e. any operator in one set commutes with all the operators in another set), one easily obtains

$$b_{2m}(t) = b_{2m}(0)\,\mathrm{e}^{-\mathrm{i}2m\bar{\Omega}t} \qquad b_{2m}^\dagger(t) = b_{2m}^\dagger(0)\,\mathrm{e}^{\mathrm{i}2m\bar{\Omega}t} \tag{4.1a}$$

$$c_{2n+1}(t) = c_{2n+1}(0)\,\mathrm{e}^{-\mathrm{i}\lambda_{2n+1}\bar{\Omega}t} \qquad c_{2n+1}^\dagger(t) = c_{2n+1}^\dagger(0)\,\mathrm{e}^{\mathrm{i}\lambda_{2n+1}\bar{\Omega}t} \tag{4.1b}$$

where $\bar{\Omega} = \Omega\sqrt{1 - 4\epsilon^2}$, $\lambda_{2n+1} = (2n+1), c_{2n+1} = b_{2n+1}$ as $n \geqslant k$, while $\lambda_{2n+1}, n = 0, 1, \ldots, (k-1)$ are determined from equation (3.11), and the relation between $\{c_{2n+1}, c_{2n+1}^\dagger, n = 0, 1, \ldots, (k-1)\}$ and $\{b_{2n+1}, b_{2n+1}^\dagger, n = 0, 1, \ldots, (k-1)\}$ is given by equation (3.13).

Our purpose in this section is to find explicitly how the operators of the radiation field vary with respect to time for the given initial bare operators $a_m^\dagger(0)$ and $a_m(0)$, $m = 1, 2, 3, \ldots$. Equations (2.7) and (4.1) result in

$$b_{2m}(t) = \left[\sum_{n=1}^{\infty} U_{mn}(-\theta) a_{2n}(0)\right] \mathrm{e}^{-\mathrm{i}2m\bar{\Omega}t} \qquad m = 1, 2, 3, \ldots \tag{4.2a}$$

$$a_{2n}(t) = \sum_{m=1}^{\infty} G_{nm}(\theta, t)\, a_{2m}(0) \qquad n = 1, 2, 3, \ldots \tag{4.2b}$$

where $U_{nm}(\theta) = \langle n| \exp[\theta(\sqrt{N}A - A^\dagger\sqrt{N})]|m\rangle$, $\theta = \tanh^{-1}(2\epsilon)/2$, and the matrix elements

$$G_{nm}(\theta, t) = \sum_{k=1}^{\infty} U_{nk}(\theta)U_{km}(-\theta)\exp(-\mathrm{i}2k\bar{\Omega}t)$$

which can easily be expressed, just as we did in the first resonance case [13], as follows:

$$G_{nm}(\theta, t) = \sqrt{mn}\left(\frac{1-\mathrm{i}q}{1+\mathrm{i}q}\right)^n \left(\frac{if}{1+\mathrm{i}q}\right)^{n+m-2} \times \sum_{k=0}^{\min(n,m)} (-1)^{m-k}\frac{(n+m-k-1)!}{k!(m-k)!(n-k)!}\left(\frac{f^2}{1+q^2}\right)^{n-k} \tag{4.3}$$

where $f = \sinh(2\theta)\tan(\bar{\Omega}t)$ and $q = \cosh(2\theta)\tan(\bar{\Omega}t)$. We have now expressed exactly all the time-varying annihilation operators of the quantized field with even subscripts explicitly in terms of the corresponding initial bare operators of the field.

Using equations (2.7b), (3.12) and (4.1), and the fact that the transformation matrices P, Q and V are all real matrices, we can obtain

$$c_{2n+1}(t) = \left[\sum_{l=0}^{\infty} V_{ln}(\phi)\, a_{2l+1}(0)\right]\mathrm{e}^{-\mathrm{i}(2n+1)\bar{\Omega}t} \qquad n \geqslant k \tag{4.4a}$$

$$c_{2n+1}(t) = \left\{\sum_{l=0}^{\infty}[\sigma_{nl}a_{2l+1}(0) + \tau_{nl}a^\dagger_{2l+1}(0)]\right\}\mathrm{e}^{-\mathrm{i}\lambda_{2n+1}\bar{\Omega}t} \qquad 0 \leqslant n \leqslant k-1 \tag{4.4b}$$

$$a_{2n+1}(t) = \sum_{l=0}^{\infty}\left[g_{nl}(t)\, a_{2l+1}(0) + w_{nl}(t)\, a^\dagger_{2l+1}(0)\right] \qquad n = 0, 1, 2, \ldots \tag{4.5}$$

where $V_{jl}(-\phi) = V_{lj}(\phi)$ is used, and

$$\sigma_{nl} = \sum_{m=0}^{k-1} P_{nm}V_{lm}(\phi) \qquad \tau_{nl} = \sum_{m=0}^{k-1} Q_{nm}V_{lm}(\phi) \tag{4.6}$$

$$g_{nl}(t) = \langle n|V(\phi)\exp(-\mathrm{i}(2N+1)\bar{\Omega}t)V^{-1}(\phi)|l\rangle + \sum_{m,j=0}^{k-1} V_{nm}(\phi)\left[P^T D(t)P - Q^T D^*(t)Q - D^{(0)}(t)\right]_{mj}V_{lj}(\phi) \tag{4.7a}$$

$$w_{nl}(t) = \sum_{m,j=0}^{k-1} V_{nm}(\phi)\left[P^T D(t)Q - Q^T D^*(t)P\right]_{mj}V_{lj}(\phi) \tag{4.7b}$$

where $D(t)$ and $D^{(0)}(t)$ are two $k \times k$ diagonal matrices with the diagonal elements $D_{nn}(t) = \exp(-\mathrm{i}\lambda_{2n+1}\bar{\Omega}t)$ and $D^{(0)}_{nn}(t) = \exp[-(2n+1)i\bar{\Omega}t]$ $(n = 0, 1, \ldots, k-1)$. Noting $D(0) = D^{(0)}(0) = I$ and using equation (3.7), we see $g_{nl}(0) = \delta_{nl}$, $w_{nl}(0) = 0$, and hence the left-hand side of equation (4.5) equals $a_{2n+1}(0)$ at time $t = 0$ as it should be. In equations (4.4)–(4.7), $\bar{\Omega} = \Omega\sqrt{1-4\epsilon^2}$, $V_{nl}(\phi) = \langle n|V(\phi)|l\rangle$ with $V(\phi)$ given by equation (2.5), the matrices P and Q as well as λ_{2n+1}, $n = 0, 1, \ldots, (k-1)$ are determined by equation (3.11). Since the time evolution expressions of the annihilation operators is obtained, we can calculate the photon creations out of the vacuum due to the oscillating boundary. Let $|vac\rangle$ be the vacuum state of the initial bare operators, i.e. $a_n(0)|vac\rangle = 0$, $n = 1, 2, \ldots$, and $n_m(t) \equiv \langle vac|a^\dagger_m(t)\, a_m(t)|vac\rangle$, we then, from equations (4.2), and (4.5), obtain

$$n_{2l}(t) = 0 \qquad l = 0, 1, 2, \ldots \tag{4.8}$$

$$n_{2m+1}(t) = \sum_{l=0}^{\infty}|w_{ml}(t)|^2 = \langle\Psi_m|\Psi_m\rangle \qquad m = 0, 1, 2, \ldots \tag{4.9}$$

where

$$\langle\Psi_m| = (|\Psi_m\rangle)^\dagger = (V_{m,0}, V_{m,1}, V_{m,2}, \ldots, V_{m,k-1})[P^T D(t) Q - Q^T D^*(t) P] \tag{4.10}$$

and $V_{m,l} \equiv V_{ml}(\phi)$. Note that equation (4.9) manifests that the photon numbers of the bare modes $(2m+1)$ are quasi-periodic functions of the time t. For instance, utilizing equations (2.5), (B.15) and (4.9), and neglecting the terms of order equal to and greater than $\mathrm{O}(\epsilon^5)$, we obtain $n_{2m+1}(t) \approx 0, m \geqslant 2$ and

$$n_1(t) \approx \epsilon^2(4 + \tfrac{29}{6}\epsilon^2)\sin^2(\lambda_1\bar{\Omega}t) + \tfrac{3}{4}\epsilon^4\{[\cos(\lambda_1\bar{\Omega}t) - \cos(\lambda_3\bar{\Omega}t)]^2 + [3\sin(\lambda_1\bar{\Omega}t) - \sin(\lambda_3\bar{\Omega}t)]^2\} \tag{4.11a}$$

$$n_3(t) \approx \tfrac{3}{16}\epsilon^4\{[\cos(\lambda_1\bar{\Omega}t) - \cos(\lambda_3\bar{\Omega}t)]^2 + [5\sin(\lambda_1\bar{\Omega}t) + \sin(\lambda_3\bar{\Omega}t)]^2\} \tag{4.11b}$$

where $\lambda_1\bar{\Omega} = \lambda_1\Omega\sqrt{1-4\epsilon^2} \approx \Omega\sqrt{1-8\epsilon^2+3\epsilon^4} + \mathrm{O}(\epsilon^6)$ and $\lambda_3\bar{\Omega} \approx 3\Omega\sqrt{1-4\epsilon^2-\epsilon^4/2} + \mathrm{O}(\epsilon^6)$. Using $\Omega = \omega I_0(q_0/L), \omega = \pi/L$ and $\epsilon = q_0/[4LI_0(q_0/L)]$, we see that for sufficiently small time t, the photon number in the fundamental mode becomes $n_1(t) = [(q_0/L)\omega t]^2/4 + \mathrm{O}[(q_0/L)^4]$ which is, apart from the different notation, identical to equation (6.6) of Dodonov and Klimov [5] (first reference) by noting the trajectory of the moving mirror $q(t) = L\exp[q_0\cos(2\omega t)/L] = L[1 + (q_0/L)\cos(2\omega t)] + \mathrm{O}[(q_0/L)^2]$. However, our result displays that the photon numbers are quasi-periodic functions of the time t, while theirs show a monotonically increasing behaviour with respect to time t in the long-time asymptotical behaviour. This discrepancy may originate from the fact that they have, from the very beginning, neglected (in the equations governing the evolution of the amplitudes of the modes) all the terms greater than or equal to the second order of the oscillating amplitude of the moving mirror, which are important in the long-time asymptotic limits. In addition, their results are invalid when the time t does not satisfy the condition $\epsilon^2\omega t \ll 1$ [14].

Equations (4.2), (4.4) and (4.5) are the time evolution expressions of the dressed and bare annihilation operators in terms of the initial bare operators. Besides the dynamics of the quantized modes of the field, these results also completely determine its statistical properties such as various inter- and intra-modes correlations as the functions of the corresponding initial values. The total Hamiltonian is a diagonal form in terms of the two independent operator sets $\{b_{2n}, b_{2n}^\dagger; n = 1, 2, \ldots\}$ and $\{c_{2m+1}, c_{2m+1}^\dagger; m = 0, 1, \ldots\}$ (i.e. any operator in one set commutes with all the operators in another set) which are the final dressed operators dressed by the oscillations of the moving boundary. It is pointed out that equations (4.2) and (4.8) are *exact* and explicit results, while equations (4.4), (4.5) and (4.9) are only accurate up to $\mathrm{O}(\epsilon^k)$ although here k can be any positive integer, and they are explicit results only when the matrices P and Q as well as $\lambda_{2n+1}, n = 0, 1, \ldots, (k-1)$ are calculated by solving equation (3.11). In appendix B, we have provided a systematic procedure to solve equation (3.11) for any positive integer k, and have explicitly presented the matrices P and Q as well as $\lambda_{2n+1}, n = 0, 1, \ldots, (k-1)$ for $k = 4$ in equations (B.13) and (B.14), respectively, see also equation (B.15). Substituting these explicit expressions of matrices P and Q as well as the eigenvalues into equations (4.4) and (4.5), we then obtain the corresponding explicit results accurate up to $\mathrm{O}(\epsilon^4)$ on the time evolution of the bare and dressed modes with odd subscripts.

It is worthwhile to mention two interesting features in the second resonance case. These two similar features also exist in other resonance cases as well. The first one is that the parametric oscillator part $H^{(p)}$ does not affect modes with even subscripts and the effect of the oscillating boundary is to cause photon scattering solely among these modes. Consequently, the photon numbers of these modes remain zero if all of them are zero initially. In the jth resonance case, a similar conclusion is true for modes with subscripts nj ($n = 1, 2, \ldots$). The second feature is that in the vacuum state $|vac\rangle$ of the initial bare photonic operators, the photon numbers of the modes with odd subscripts, if not identically zero, are quasi-periodic functions of the time t up to any order of the small parameter ϵ. This is clearly and explicitly reflected in equation (4.9) accurate up to $O(\epsilon^k)$, where k can be an any positive integer. This conclusion is also true for other harmonic resonance cases and can be understood physically by noting the fact that the total Hamiltonian in any harmonic resonance case is in the bilinear form in terms of the bare annihilation and creation operators, and hence could be, *in principle*, diagonalized by introducing the dressed annihilation and creation operators through a *linear* unitary transformation. The dressed operators have a purely sinusoidal time dependence which, through the *linear* transformation, leads to the quasi-periodic time dependence of the bare annihilation and creation operators as well as the bare photon numbers. This implies that photon creation out of the vacuum in the oscillating cavity is not a monotonically increasing behaviour with respect to time t within the framework of the bilinear Hamiltonian derived under the rotating-wave approximation.

5. Conclusions and discussions

In summary, we have investigated the dynamics of the quantized radiation field in a one-dimensional cavity when one of its boundary oscillates with the jth eigenfrequency of the unperturbed cavity ($j = 1, 2, 3, \ldots$) by using the effective resonance Hamiltonian under the rotating-wave approximation which has the form $H = H' + \bar{H}^{(p)}$. It has been shown that the Hamiltonian $H' = \Omega \sum_{k=1}^{\infty} k a_k^\dagger a_k + \epsilon\Omega \sum_{k=1}^{\infty} \sqrt{k(k+j)}\,[a_k^\dagger a_{k+j} + a_{k+j}^\dagger a_k]$ with $j = 1, 2, 3, \ldots$ can be diagonalized exactly by introducing a novel diagonalization method. The total Hamiltonian $H = H' + \bar{H}^{(p)}$ after this diagonalization procedure has become a form suitable for a perturbative treatment, while its original form is not since the infinite terms have an equal order measured by the small oscillating amplitude. A systematic method has been developed to diagonalize the total Hamiltonian $H = H' + \bar{H}^{(p)}$ and the corresponding dynamics of the quantized electromagnetic fields within the oscillating cavity to any desired order of the small parameter ϵ characterizing the oscillating amplitude of the moving boundary of the cavity.

We have focused on the second resonance case to illustrate the systematic method. In this case, we have derived the analytical expressions of the diagonalized Hamiltonians, the time-varying annihilation and creation operators as well as photon number operators for the radiative field up to order ϵ^k for an arbitrary positive integer k in terms of the quantities to be determined by solving an eigenvalue problem of a $k \times k$ matrix. In addition, we have explicitly obtained the quantities needed in the above-mentioned analytical expressions up to order ϵ^4. In addition, there exist two additional conclusions based on the effective Hamiltonian derived under the rotating-wave approximation. First, in the jth resonance case, the parametric oscillator part $H^{(p)}$ does not affect modes with subscripts nj ($n = 1, 2, \ldots$) and the effect of the oscillating boundary on these modes is to cause photon scattering solely among these modes. Consequently, the photon numbers of these modes remain zero if all of them have no photon initially. In other words, the oscillating boundary is unable to create photon of these modes out of the vacuum state or the states in which there exists no photon in each of these modes. The second conclusion is as follows. The oscillating boundary can cause the creation of photons out of the vacuum in those modes whose subscript is not equal to nj, but the photon numbers of these modes with the zero initial value evolve as, if not identically zero, quasi-periodic functions of the time t. This conclusion implies that photon creation out of the vacuum in the oscillating cavity is not a monotonically increasing behaviour with respect to time t within the framework of the Hamiltonian derived under the rotating-wave approximation.

Lastly, we point out that besides the dynamics of the quantized modes of the field in a one-dimensional cavity when one of its boundary oscillates with the jth eigenfrequency of the unperturbed cavity, our results also completely determine the corresponding statistical properties of the field such as various inter- and intra-modes correlations as functions of the corresponding initial values. In addition the present results, together with our theory on the atom–light interaction processes in cavities with fixed boundaries [16], provide a convenient basis for dealing with microcavity-modified atom–light interaction processes in a one-dimensional microcavity when one of its boundary oscillates with the jth eigenfrequency of the unperturbed cavity.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China under grant nos 69688004 and 69788002, and the National Laboratory of MRAMP at Wuhan Institute of Physics and Mathematics, the Chinese Academy of Sciences.

Appendix A

In this appendix, we prove the relations

$$\sqrt{1-4\epsilon^2}\,U^{(k)}(\psi_j)(Nj+k)U^{(k)}(-\psi_j) = [(Nj+k)+\epsilon(f_N^{j;k}A + A^\dagger f_N^{j;k})] \tag{A.1}$$

where $f_N^{j;k} = \sqrt{(Nj+k)[(N+1)j+k]/(N+1)}$, $\psi_j = \tanh^{-1}(2\epsilon)/2j$ and $U^{(k)}(\psi_j) = \exp[\psi_j(f_N^{j;k}A - A^\dagger f_N^{j;k})]$, $k = 1, 2, \ldots, (j-1)$. In addition, we also illustrate that both B_{nm} in equation (3.1) and E_{nm} in equation (3.2) are $\mathrm{O}(\epsilon^{n+m})$ quantities.

Let $W_\pm = (f_N^{j;k}A \pm A^\dagger f_N^{j;k})$, one can easily show that

$$\begin{aligned} &\mathrm{d}[\exp(\psi_j W_-)(Nj+k)\exp(-\psi_j W_-)]/\mathrm{d}\psi_j = j\exp(\psi_j W_-)W_+\exp(-\psi_j W_-)\\ &\mathrm{d}^2[\exp(\psi_j W_-)(Nj+k)\exp(-\psi_j W_-)]/\mathrm{d}\psi_j^2 = 4j\exp(\psi_j W_-)(Nj+k)\exp(-\psi_j W_-). \end{aligned} \tag{A.2}$$

These two equations combine to give the relation

$$\exp(\psi_j W_-)(Nj+k)\exp(-\psi_j W_-) = (Nj+k)\cosh(2j\psi_j) + W_+\sinh(2j\psi_j)/2$$

which, by using $[\cosh(2j\psi_j)]^{-1} = \sqrt{1-\tanh^2(2j\psi_j)}$, becomes

$$\sqrt{1-\tanh^2(2j\psi_j)}\exp(\psi_j W_-)(Nj+k)\exp(-\psi_j W_-) = (Nj+k) + W_+\tanh(2j\psi_j)/2.$$

Taking $\tanh(2j\psi_j) = 2\epsilon$, we arrive at equation (A.1).

Equation (2.5) gives

$$V(\phi) = \exp(\phi_t W) = I + \sum_{n=1}^{\infty}\frac{\phi^n W^n}{n!} \tag{A.3}$$

where $W = f_N A - A^\dagger f_N$, $f_N = \sqrt{(2N+1)(2N+3)/(N+1)}$, and $\phi = \tanh^{-1}(2\epsilon)/4 = \mathrm{O}(\epsilon)$. It is easily shown that $\langle 0|W^n|m\rangle = 0$ for $0 < n < m$, and $\langle 0|W^m|m\rangle = \langle 0|(f_N A)^n|m\rangle = \sqrt{m!}f_0 f_1 \ldots f_{m-1}$ for $m > 0$. Therefore, $\phi_m \equiv \langle 0|V(\phi)|m\rangle = \mathrm{O}(\phi^m) = \mathrm{O}(\epsilon^m)$.

Consequently, $B_{nm} = \phi_n\phi_m = \mathrm{O}(\epsilon^{m+n})$. Similarly, we can show $U^{(k)}_{0m}(\psi_3) = \mathrm{O}(\psi_3^m) = \mathrm{O}(\epsilon^m)$ and thus $E_{nm} = U^{(1)}_{0n}(\psi_3)U^{(2)}_{0m}(\psi_3) = \mathrm{O}(\epsilon^{m+n})$.

Now we give some explicit results on $\phi_m = \langle 0|V(\phi)|m\rangle = \langle m|V(-\phi)|0\rangle$ which will be used in appendix B. Using $4\phi = \tanh^{-1}(2\epsilon) = \sum_{n=0}(2\epsilon)^{2n+1}/(2n+1)$ and $\langle m|W|n\rangle = g_m\delta_{n,m+1} - g_{m-1}\delta_{n,m-1}$ with $g_m = \sqrt{(2m+1)(2m+3)}$ and some manipulations, we can obtain

$$\phi_0 = 1 - \tfrac{3}{2}\left(\tfrac{1}{2}\epsilon\right)^2 - \left(15 + \tfrac{5}{8}\right)\left(\tfrac{1}{2}\epsilon\right)^4 - \left(190 + \tfrac{163}{240}\right)\left(\tfrac{1}{2}\epsilon\right)^6 + \mathrm{O}(\epsilon^8) \tag{A.4a}$$

$$\phi_1 = \tfrac{1}{2}\sqrt{3}\epsilon\left[1 + \left(4 + \tfrac{5}{6}\right)\left(\tfrac{1}{2}\epsilon\right)^2 + \left(48 + \tfrac{1}{40}\right)\left(\tfrac{1}{2}\epsilon\right)^4\right] + \mathrm{O}(\epsilon^7) \tag{A.4b}$$

$$\phi_2 = \tfrac{3}{2}\sqrt{5}\left(\tfrac{1}{2}\epsilon\right)^2\left[1 + \tfrac{15}{2}\left(\tfrac{1}{2}\epsilon\right)^2 + \left(73 + \tfrac{27}{36}\right)\left(\tfrac{1}{2}\epsilon\right)^4\right] + \mathrm{O}(\epsilon^8) \tag{A.4c}$$

$$\phi_3 = \sqrt{7}\left(\tfrac{1}{2}\epsilon\right)^3\left[\tfrac{5}{2} + \left(27 + \tfrac{3}{8}\right)\left(\tfrac{1}{2}\epsilon\right)^2\right] + \mathrm{O}(\epsilon^7) \tag{A.4d}$$

$$\phi_4 = \tfrac{105}{8}\left(\tfrac{1}{2}\epsilon\right)^4 + 105\left(\tfrac{1}{2}\epsilon\right)^6 + \mathrm{O}(\epsilon^8). \tag{A.4e}$$

Appendix B

In this appendix, we develop a systematic procedure to solve the eigenvalue equation in equation (3.11), i.e. $R_+^{-1}WR_+ = \Lambda^2$ where $W = (F+\eta B)(F-\eta B) = F^2 - 2\eta[N, B] - \eta^2 B$, $F = 2N+1$, $F_{nm} = (2n+1)\delta_{nm}$, $B_{nm} = \phi_n\phi_m$ and $\phi_m = \langle 0|V(\phi)|m\rangle = \langle m|V(-\phi)|0\rangle$ are real numbers. In writing W, we have made use of the property $B^2 = B$. This can be easily proved by noting that $B^2_{nm} = \sum_j B_{nj}B_{jm} = \phi_n\phi_m\sum_j\phi_j^2$ and $\sum_j\phi_j^2 = \sum_j\langle 0|V(\phi)|j\rangle\langle j|V(-\phi)|0\rangle = \langle 0|V(\phi)V(-\phi)|0\rangle = \langle 0|0\rangle = 1$.

To develop the systematic procedure to solve $R_+^{-1}WR_+ = \Lambda^2$, we frequently need to use the identity

$$\mathrm{e}^{\eta S}D\mathrm{e}^{-\eta S} = D + \sum_{k=1}^{\infty}\frac{\eta^k}{k!}G_k \tag{B.1}$$

where $G_k = [S, G_{k-1}]$, $G_0 = D$ and $[A, B] = AB - BA$. Let D is a diagonal matrix with $D_{nm} = (2n+1)^2\delta_{nm}$, $S = \sum_{j=0}^{\infty}\eta^j S^{(j)}$, and $[S^{(0)}, D] = -2[N, B]$, we obtain

$$W = \mathrm{e}^{\eta S}D\mathrm{e}^{-\eta S} - \eta^2 K - \sum_{k=2}^{\infty}\frac{\eta^k}{k!}G_k \tag{B.2}$$

where $K = B + \left[\sum_{j=1}^{\infty}\eta^{j-1}S^{(j)}, D\right]$. Using

$$\mathrm{e}^{-\eta S}K\mathrm{e}^{\eta S} = \sum_{l=0}^{\infty}\frac{(-\eta)^l}{l!}K_l \tag{B.3}$$

where $K_l = [S, G_{l-1}]$ and $G_0 = K$, and

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{\eta^k}{k!}\mathrm{e}^{-\eta S}G_k\mathrm{e}^{\eta S} &= \sum_{k=2}^{\infty}\frac{(\eta)^k}{k!}\sum_{m=0}^{\infty}\frac{(-\eta)^m}{m!}G_{k+m}\\ &= \sum_{n=2}^{\infty}(-\eta)^n G_n\sum_{k=2}^{n}(-1)^k\frac{1}{k!(n-k)!}\\ &= \sum_{n=2}^{\infty}\frac{(-\eta)^n}{n(n-2)!}G_n\end{aligned} \tag{B.4}$$

we then yield

$$e^{-\eta S} W e^{\eta S} = D - \eta^2 \sum_{l=0}^{\infty} \frac{(-\eta)^l}{l!} K_l - \sum_{n=2}^{\infty} \frac{(-\eta)^n}{n(n-2)!} G_n. \tag{B.5}$$

Our purpose is to choose the matrices $S^{(j)}$ in $S = \sum_{j=0}^{\infty} \eta^j S^{(j)}$ such that the right-hand side of this equation becomes a diagonal matrix $\Lambda^2 = \mathrm{diag}(\lambda_1^2, \lambda_3^2, \ldots, \lambda_{2n+1}^2, \ldots)$ and hence $R_+ = \exp(\eta S)$. This can be done by expanding the right-hand side of this equation into the power series of the variable η as follows:

$$e^{-\eta S} W e^{\eta S} = D + \eta^2 X + \eta^3 Y + \sum_{l=4}^{\infty} \eta^l Z^{(l)} \tag{B.6}$$

and requiring that the coefficient matrices X, Y and $Z^{(l)}, l = 4, 5, \ldots$ become diagonal matrices. It is pointed out that $(S^{(2j)})^T = S^{(2j)}, (S^{(2j+1)})^T = -S^{(2j+1)}, j = 0, 1, 2, \ldots$ in order to satisfy $R_- = R_+|_{\eta\to-\eta}$ and $R_-^T R_+ = I$. It can then be shown that $Y^T = -Y$ and $(Z^{(2j+1)})^T = -Z^{(2j+1)}$ $j = 0, 1, \ldots$ and hence $Y_{nn} = Z_{nn}^{(2j+1)} = 0$. Therefore, we have

$$\lambda_{2n+1}^2 = (2n+1)^2 + \eta^2 X_{nn} + \sum_{l=2}^{\infty} \eta^{2l} Z_{nn}^{(2l)} \tag{B.7}$$

where $n = 0, 1, \ldots$. We have finished describing the outline of the systematic procedure.

Now, we derive explicit expressions of the eigenvalues and the transformation matrices P and Q accurate up to $O(\epsilon^4)$ by this systematic procedure. From equation (B.5), one can obtain

$$X = -B - [S^{(1)}, D] + [S^{(0)}, [N, B]] \tag{B.8}$$

$$Y = [S^{(0)}, B] - [S^{(2)}, D] + [S^{(0)}, [S^{(1)}, D]] + [S^{(1)}, [N, B]] - \tfrac{1}{2}[S^{(0)}, [S^{(2)}, D]] + \tfrac{2}{3}[S^{(0)}, [S^{(0)}, [N, B]]] \tag{B.9}$$

where $D = (2N+1)^2$ $(D_{nm} = (2n+1)^2 \delta_{nm})$, and use has been made of the relation $[S^{(0)}, D] = -2[N, B]$, or

$$S_{nm}^{(0)} = \frac{\phi_n \phi_m}{2(n+m+1)}. \tag{B.10}$$

Equation (B.8) displays that X becomes a diagonal matrix if we choose

$$S_{nm}^{(1)} = \frac{\phi_n \phi_m}{4(n-m)(n+m+1)} \left[1 + \frac{1}{2} \sum_{l=0}^{\infty} \phi_l^2 \left(\frac{m-l}{n+l+1} + \frac{n-l}{m+l+1}\right)\right](1 - \delta_{nm}) \tag{B.11}$$

and hence

$$X_{nn} = -\phi_n^2 \left[1 + \sum_{l=0}^{\infty} \frac{(n-l)\phi_l^2}{n+l+1}\right]. \tag{B.12}$$

It is noted that $(S^{(0)})^T = S^{(0)}$ and $(S^{(1)})^T = -S^{(1)}$ as they should be. Using equations (B.9)–(B.11) and the facts $\phi_m = O(\epsilon^m)$, $B_{nm} = \phi_n \phi_m = O(\epsilon^{n+m})$, and $\eta = O(\epsilon)$, it is easily seen that $R_+ = \exp(\eta S^{(0)} + \eta^2 S^{(1)}) + O(\epsilon^5)$, $R_- = R_+|_{\eta\to-\eta} = \exp(-\eta S^{(0)} + \eta^2 S^{(1)}) + O(\epsilon^5)$ and $\lambda_{2n+1}^2 = (2n+1)^2 + \eta^2 X_{nn} + O(\epsilon^6)$. Using $\eta = 2\epsilon/\sqrt{1-4\epsilon^2}$, and equations (A.4), (B.6) and (B.11), (B.12), after some manipulations we obtain the eigenvalues

$$\lambda_1 = \sqrt{1 - 4\epsilon^2 - 13\epsilon^4} + O(\epsilon^6) \tag{B.13a}$$

$$\lambda_3 = 3\sqrt{1 - \epsilon^4/2} + O(\epsilon^6) \tag{B.13b}$$

$$\lambda_{2n+1} = (2n+1) + O(\epsilon^6) \qquad \text{as} \quad n \geqslant 2 \tag{B.13c}$$

and the transformation matrices $P^T = (R_+ + R_-)/2$ and $Q^T = (R_+ - R_-)/2$ needed in equation (3.12) as follows:

$$P^T = \begin{pmatrix} 1 + \frac{1}{2}\epsilon^2 + \frac{133}{96}\epsilon^4 & -\frac{1}{4}\sqrt{3}\epsilon^3 & -\frac{1}{16}\sqrt{5}\epsilon^4 & 0 \\ \frac{1}{2}\sqrt{3}\epsilon^3 & 1 + \frac{3}{32}\epsilon^4 & 0 & 0 \\ \frac{3}{16}\sqrt{5}\epsilon^4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathrm{O}\left(\epsilon^5\right) \tag{B.14a}$$

$$Q^T = \begin{pmatrix} \epsilon + \frac{17}{12}\epsilon^3 & \frac{1}{4}\sqrt{3}\epsilon^2\left(1 + \frac{19}{12}\epsilon^2\right) & \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{5}{64}\sqrt{7}\epsilon^4 \\ \frac{1}{4}\sqrt{3}\epsilon^2\left(1 + \frac{49}{12}\epsilon^2\right) & \frac{1}{4}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 \\ \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 & 0 \\ \frac{5}{64}\sqrt{7}\epsilon^4 & 0 & 0 & 0 \end{pmatrix} + \mathrm{O}\left(\epsilon^5\right). \tag{B.14b}$$

Consequently, equation (3.12) in this case becomes

$$\begin{pmatrix} c_1 \\ c_3 \\ c_5 \\ c_7 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\epsilon^2 + \frac{133}{96}\epsilon^4 & \frac{1}{2}\sqrt{3}\epsilon^3 & \frac{3}{16}\sqrt{5}\epsilon^4 & 0 \\ -\frac{1}{4}\sqrt{3}\epsilon^3 & 1 + \frac{3}{32}\epsilon^4 & 0 & 0 \\ -\frac{1}{16}\sqrt{5}\epsilon^4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_3 \\ b_5 \\ b_7 \end{pmatrix}$$
$$+ \begin{pmatrix} \epsilon + \frac{17}{12}\epsilon^3 & \frac{1}{4}\sqrt{3}\epsilon^2(1 + \frac{49}{12}\epsilon^2) & \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{5}{64}\sqrt{7}\epsilon^4 \\ \frac{1}{4}\sqrt{3}\epsilon^2(1 + \frac{19}{12}\epsilon^2) & \frac{1}{4}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 \\ \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 & 0 \\ \frac{5}{64}\sqrt{7}\epsilon^4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1^\dagger \\ b_3^\dagger \\ b_5^\dagger \\ b_7^\dagger \end{pmatrix}$$
$$+\mathrm{O}\left(\epsilon^5\right) \tag{B.15a}$$

$$\begin{pmatrix} b_1 \\ b_3 \\ b_5 \\ b_7 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\epsilon^2 + \frac{133}{96}\epsilon^4 & -\frac{1}{4}\sqrt{3}\epsilon^3 & -\frac{1}{16}\sqrt{5}\epsilon^4 & 0 \\ \frac{1}{2}\sqrt{3}\epsilon^3 & 1 + \frac{3}{32}\epsilon^4 & 0 & 0 \\ \frac{3}{16}\sqrt{5}\epsilon^4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \\ c_5 \\ c_7 \end{pmatrix}$$
$$- \begin{pmatrix} \epsilon + \frac{17}{12}\epsilon^3 & \frac{1}{4}\sqrt{3}\epsilon^2(1 + \frac{19}{12}\epsilon^2) & \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{5}{64}\sqrt{7}\epsilon^4 \\ \frac{1}{4}\sqrt{3}\epsilon^2(1 + \frac{49}{12}\epsilon^2) & \frac{1}{4}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 \\ \frac{1}{8}\sqrt{5}\epsilon^3 & \frac{3}{64}\sqrt{15}\epsilon^4 & 0 & 0 \\ \frac{5}{64}\sqrt{7}\epsilon^4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1^\dagger \\ c_3^\dagger \\ c_5^\dagger \\ c_7^\dagger \end{pmatrix}$$
$$+\mathrm{O}\left(\epsilon^5\right). \tag{B.15b}$$

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